Šta sve možemo sa slučajnim tačkama u prostoru?

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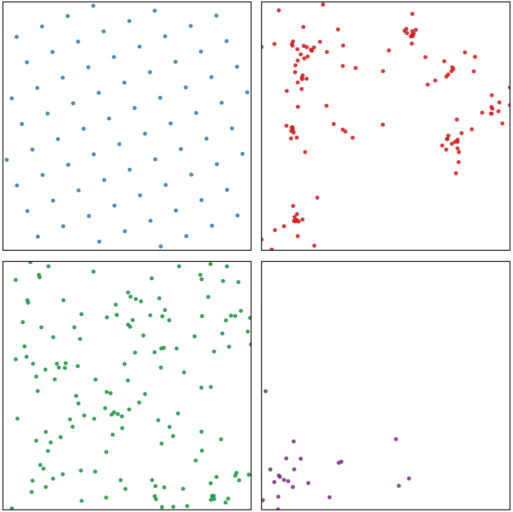
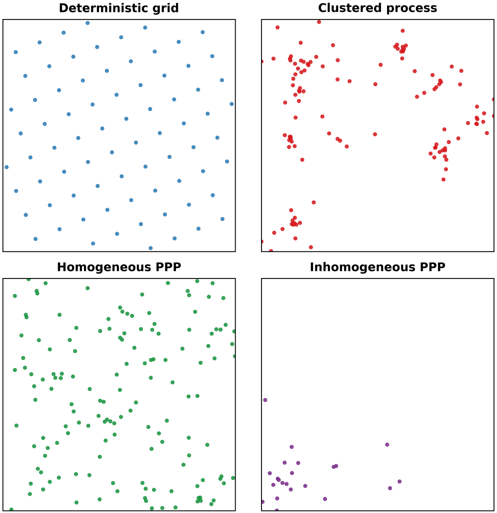
Šta mislimo pod "slučajnim" tačkama u ravni?



What does “random” look like?



Poisson Point Process

- Point processes - special type of stochastic processes
- A **Poisson point process**(PPP) N on $\mathbb{R}^d$ with intensity $\lambda > 0$:

1. The number of points in any bounded region A has Poisson distribution:

$$N(A) \sim \mathrm{Poisson}(\lambda|A|)$$

2. Given $N(A) = n$, the n points are **uniformly distributed** in A
3. Counts over **disjoint** regions are **independent**

Key properties

Superposition of two independent PPPs is a PPP.
Random thinning of a PPP is a PPP.

Inhomogeneous PPP

Replace the constant λ with an **intensity function** $\lambda(\mathbf{x})$:

$$\mathbb{E}[N(A)] = \int_A \lambda(\mathbf{x})\, d\mathbf{x}$$

Homogeneous

Constant density everywhere

$\lambda(\mathbf{x}) = \lambda$

Inhomogeneous

Density varies in space

$\lambda(\mathbf{x})$ – any non-negative, locally integrable function

Independence and Poisson counts still hold – only the intensity changes.

Where Do Random Points Appear?

Natural phenomena

- Seeds dispersed by a plant
- Raindrops on a surface
- Locations of trees in a forest

Biology / Ecology

- Spatial distribution of species
- Island biogeography models

Telecommunications

- Arrivals of phone calls
- Base station locations
- Loss networks

Engineering

- Sensor placement and coverage
- Wireless network connectivity

Seed Dispersal – The Model

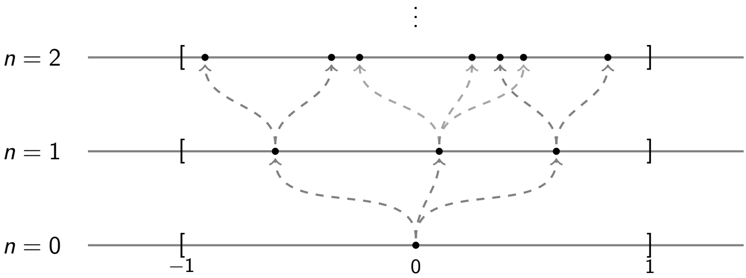
Biological motivation: Seeds are dispersed around the mother plant by wind, birds, water, . . . in a random manner.

- Time is discrete, scaled by plant generations
- Habitat $=$ an *island* $[-L, L]$ (mountaintop, oasis, grassland surrounded by houses, . . .)
- One plant at origin; offspring landing sites $\sim \mathrm{PPP}(\lambda/2)$ on $[-1, 1]$
- Each offspring repeats the cycle independently
- Births outside $[-L, L]$ are lost

This is a **Branching Random Walk (BRW)** with two barriers.

Key question: Does the species survive or go extinct? [Coletti, M., Rodriguez 2023]

Process Dynamics



Phase Transition

A system undergoes a **phase transition** when a small change in a parameter causes a dramatic change in behavior.

Physical example:

Ice $\longrightarrow$ water $\longrightarrow$ steam

Critical parameter: temperature

Sharp transition at 0°C and 100°C

In our model:

Extinction $\longleftrightarrow$ Survival

Critical parameter: λ (reproduction rate)

Sharp transition at λ_c

$\lambda < \lambda_c$ extinction a.s.

$\lambda > \lambda_c$ survival possible

Phase Transition – Seed Dispersal

Theorem (Coletti, M., Rodriguez 2023)

There exists a critical value λ_c^L such that:

- $\lambda < \lambda_c^L$: *the population dies out almost surely*
- $\lambda > \lambda_c^L$: *survival is possible*

- Without barriers: $\lambda_c = 1$ (classical result)
- With barriers at $\pm L$: $\lambda_c^L > 1$
- As $L \to \infty$: $\lambda_c^L \to 1$

Key tool: Sandwiching by multi-type branching processes whose critical parameters are related to eigenvalues of banded Toeplitz matrices.

How Does λ_c^L Change with L?

Discrete-space version - exact result:

$$\lambda_c^L = \frac{3}{1 + 2\cos\frac{\pi}{L+1}}, \qquad L = 1, 2, \ldots$$

Continuous-space - numerical bounds: $1.286814 \leq \lambda_c^1 \leq 1.287191$

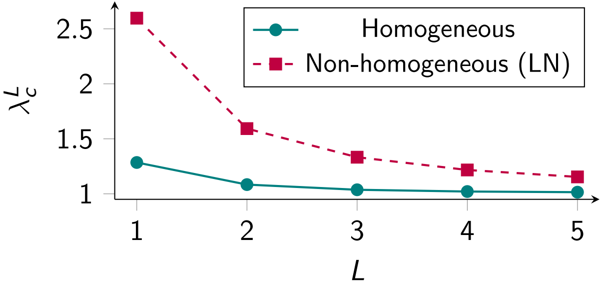
Non-homogeneous PPP: Biological argument: more seeds accumulate near the mother plant - dispersal follows a **heavy-tailed** distribution. **Extension:** Replace $\mathrm{PPP}(\lambda/2)$ with a

NHPPP driven by Log-normal density:

$$\lambda(x) = \frac{\lambda}{x\sqrt{2\pi}} \exp\left(-\frac{(\ln x)^2}{2}\right), \quad x > 0$$

Non-Homogeneous Extension

- More realistic - more seeds close to mother plant
- Simulation via thinning of a 2D homogeneous PPP
- Critical values significantly higher - fewer seeds reach the boundary

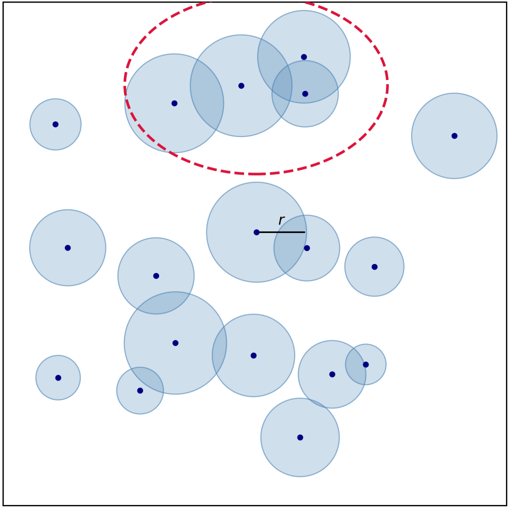


[M.]

Another model: Continuum Percolation

Construction:

1. Start with a homogeneous $\mathrm{PPP}(\lambda)$ on $\mathbb{R}^2$
2. Place a disk of radius r around each point
3. Ask: does the union of disks contain an **unbounded connected component**?



r can be random or deterministic (fixed)

Phase transition phenomenon here depends on λ and r

Theorem

There exists a critical intensity $\lambda_c(r)$ such that:

- $\lambda < \lambda_c(r)$: *all components are bounded a.s.*
- $\lambda > \lambda_c(r)$: *a unique unbounded component exists a.s.*

Coverage Processes and Sensor Networks

Related model: Place sensors at locations given by a $\mathrm{PPP}(\lambda)$. Each sensor covers a disk of radius r.

- **Coverage:** Is every point of a region covered by at least one sensor?
- **Connectivity:** Can every sensor communicate with every other?
- Both questions exhibit phase transitions in λ and r

Connection to continuum percolation: Coverage and connectivity are closely related to the existence of an unbounded connected component.

Same mathematical framework – different questions.

Loss Networks - one-dimensional continuous Model

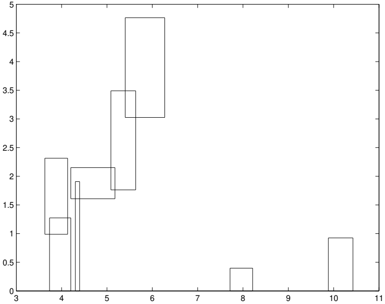
- Calls arrive as $\mathrm{PPP}(\lambda)$
- Each call has a **random length** (distribution π, moments ρ_1, ρ_2) and **exponential lifetime** (mean 1)
- Network has capacity C: a call is *lost* if C calls already overlap at that location
- State $\eta \in 1, \ldots, C^{\mathcal{B}(\mathbb{R})}$: current configuration of calls

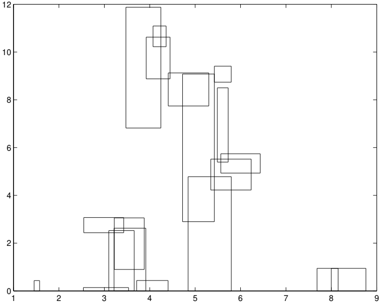
Generator of the process:

$$Af(\eta) = \int \big(f(\eta{+}\delta_\gamma) - f(\eta)\big)\, b(\gamma,\eta)\,\mathrm{d}\gamma + \int \big(f(\eta{-}\delta_\gamma) - f(\eta)\big)\eta(\mathrm{d}\gamma)$$

[Garcia, Marić]

Graphical representation

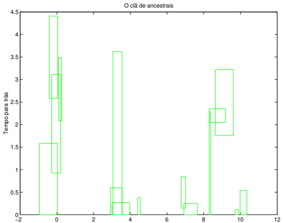


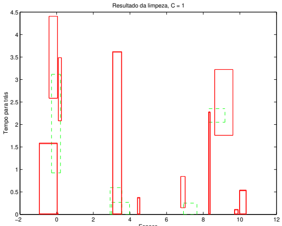


Backward Continuum Percolation - Clan of Ancestors

Key idea: To simulate the stationary distribution, look *backwards* in time.

- Each active call may have been *influenced* by earlier calls that overlapped with it
- The **clan of ancestors** = all past calls that could have affected it (directly or transitively)
- This induces a **backward continuum percolation** process
- The simulation is feasible iff the clan is **finite** a.s. $\Leftrightarrow$ percolation is **sub-critical**





Critical Value for Loss Networks

Result: The process has a unique stationary distribution whenever the backward percolation is sub-critical, which holds for:

$$\lambda < \lambda_c < \frac{1}{\rho_2 + \rho_1 + 1}$$

where ρ_1, ρ_2 are the first and second moments of the call-length distribution π.

- Improves on the earlier bound of Fernández, Ferrari and Garcia (2002)
- Obtained via domination by branching processes
- The bound depends on the second moment - not just the mean

[Garcia, Marić]

Summary

The Poisson point process is the natural model for independent random points in space - and the building block for much richer models. **A common thread: phase transitions**

- Seed dispersal on islands: survival vs. extinction
- Continuum percolation: finite vs. infinite component
- Loss networks: feasibility of perfect simulation

Open directions:

- Exact critical values - closing the gap between theoretical bounds
- Theoretical treatment of non-homogeneous models
- Higher dimensions - what changes in $\mathbb{R}^d$?
- Applications to climate change - shrinking habitats and critical thresholds for species survival

Hvala na pažnji.